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SEQUENCES WITHOUT MINIMAL SUBBASES

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Sequences without minimal subbases

by

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ABSTRACT

A sequence of cardinality z is an infinite topological space consisting of an open discrete subset of z points and a single adherence point of this subset. A subbase of a topological space is called minimal provided each proper subcollection generates a weaker topology. It is shown that for each infinite cardinal number z a sequence of cardinality z exists which allows no minimal subbase for its topology, thus answering a question posed by P. van Emde Boas.

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NOTATIONS 1. Let X be some set, and let S be a subcollection of the powerset $P(X)$. By $S^{\wedge}(S^{\vee})$ we denote the collection of all finite intersections (arbitrary unions) of members of S. The collection $(S^{\wedge})^{\vee}$ is denoted $\Gamma(S)$. By convention $\bigcup_{\emptyset} = \emptyset$ and $\bigcap_{\emptyset} = X$. The collection $\Gamma(S)$ is nothing but the topology on X which is generated by S and S is called a subbase for $\Gamma(S)$.

DEFINITION 2. Let $(X,\mathcal{O})$ be a topological space. The collection S is called a *minimal subbase* for $(X,\mathcal{O})$ provided $\Gamma(S) = \mathcal{O}$ and one has $\Gamma(S') \subsetneq \mathcal{O}$ for each proper subcollection $S' \subsetneq S$. A space $(X,\mathcal{O})$ which allows a minimal subbase for its topology is called a *subminispace*.

The concept of a minimal subbase was introduced by P. van Emde Boas [1]. We mention the following results.

(i) Every finite space is a subminispace.

(ii) The topological product of subminispaces is a subminispace.

(iii) Each metrizable space is a subminispace.

(iv) Each ordinal number (with the order topology) is a subminispace (hence the ordinary sequence itself is a subminispace).

(v) There exist normal spaces which are not subminispaces.

The spaces constructed in order to prove (v) are examples of (generalized) sequences (cf. def.3 below). In the proof in [1] the cardinality condition $\underline{cf}(z) > a$ has been used (see the definitions below), leaving as an unsolved problem what happens if this condition is not fulfilled. In this report we describe some constructions which yield a sequence which is not a subminispace for each infinite cardinal number.

We write $a,(b,c)$ for the cardinal numbers $\aleph_0,(\aleph_1,2^{\aleph_0})$.

DEFINITION 3. A *sequence of cardinality z* is a topological space $(X,\mathcal{O})$ with $|X| = z \geq a$, which consists of an open discrete set $U = X\backslash\{x_0\}$ and a single adherence point $\{x_0\}$ of U.

Clearly the topology for a sequence is fully described by presenting a neighborhood base for its unique non-isolated point x_0.

DEFINITION 4. Let x be a point in the topological space $(X,\mathcal{O})$. A *neighborhood subbase* at x is a collection $\mathcal{U}$ such that $\mathcal{U}^{\wedge}$ is a neighborhood base for x. The *local weight* at x is the minimal cardinality of a neighborhood subbase at x. Notation $\underline{\mathrm{lw}}(x)$.

Note that $\underline{\mathrm{lw}}(x)$ equals the minimal cardinality of a neighborhood base at x.

DEFINITION 5. Let z be a cardinal number. The *cofinality* of z, notation $\underline{\mathrm{cf}}(z)$, is the least cardinal number y such that z is the sum of y cardinal numbers less than z.

Clearly for infinite z one has $a \leq \underline{\mathrm{cf}}(z) \leq z$. Moreover, $\underline{\mathrm{cf}}(b) \neq a$

The technique used in order to construct sequences which allow no minimal subbase for their topology is described in the lemma below.

LEMMA 6. *Let* $(X,\mathcal{O})$ *be a sequence of cardinality* z*, with non-isolated point* x_0*. Assume, moreover, that*

(i) $\underline{\mathrm{lw}}(x_0) > z$.

(ii) *Each collection* $\mathcal{U}$ *of* $\underline{\mathrm{lw}}(x_0)$ *neighborhoods of* x_0 *contains an infinite subcollection* $\mathcal{V}$ *such that* $\cap\mathcal{V}$ *is a neighborhood of* x_0.

Then $(X,\mathcal{O})$ *is not a subminispace.*

PROOF. The proof strictly follows [1].

Let $\mathcal{S}$ be a subbase for $(X,\mathcal{O})$ and assume by hypothesis to be shown contradictory that $\mathcal{S}$ is minimal. For each singleton $\{x\}$ with $x \neq x_0$ there exists a finite collection of elements in $\mathcal{S}$ such that their intersection equals $\{x\}$. Clearly the union $\mathcal{S}'$ of all these collections has cardinality $\leq z$ and since $\mathcal{S}$ also contains a neighborhood subbase for x_0 of cardinality $\geq \underline{\mathrm{lw}}(x_0) > z$ we know that $|\mathcal{S}\backslash\mathcal{S}'| > z$.

Let $\mathcal{V} = \mathcal{S}\backslash\mathcal{S}'$. Clearly each member $V \in \mathcal{V}$ is a neighborhood of x_0, since otherwise V would be contained in $\Gamma(\mathcal{S}')$ and might be omitted from the subbase $\mathcal{S}$. Moreover, no member V of $\mathcal{V}$ contains a neighborhood of x_0 which is a finite intersection $U_1 \cap \ldots \cap U_k$ of members U_i of $\mathcal{S}$ different from V, since in this case V also could be deleted from the subbase $\mathcal{S}$. Hence we conclude that whenever $x_0 \in U \subset V$ and $U \in \mathcal{S}^{\wedge}$ the element V occurs essentially among the elements whose intersection yields U; i.e. removal of V from these elements yields an intersection which is not contained within V.

However, by assumption (ii) $\mathcal{V}$ contains an infinite subcollection $\mathcal{W}$ such that $\cap\mathcal{W}$ is a neighborhood of x_0. Writing $x_0 \in U_1 \cap \ldots \cap U_k \subseteq \cap\mathcal{W}$ we arrive at the contradiction that the infinitely many elements in $\mathcal{W}$ all are contained in the finite set $\{U_1, \ldots, U_k\}$. □

By the above lemma the problem of constructing sequences which are no subminispaces is reduced to finding neighborhood systems of x_0 satisfying (i) and (ii).

In the sequel D_z denotes a discrete space of cardinality z and $S_{z,y}$ denotes the sequence which results by adjoining a single adherence point x_0 to D_z, whose neighborhoods are all subsets of $S_{z,y}$ having a complement of cardinality $< y$. The space $S_{z,z}$ is denoted by S_z. It is easy to prove that the space S_z is a subminispace.

First we describe the construction given by P. van Emde Boas in [1].

Let W_z be the product space $S_z \times D_z$ and let X_z be the quotient space constructed from W_z by identifying $\{x_0\} \times D_z$ to a single point y_0.

<u>PROPOSITION 7</u>. *Let* $\underline{\mathrm{cf}}(z) > a$ *or let* $z = a$. *Then* X_z *satisfies* (i) *and* (ii).

<u>PROOF</u>. By the usual diagonalization argument one proves that $\underline{\mathrm{lw}}(y_0) > z$. In the case that $\underline{\mathrm{cf}}(z) > a$ condition (ii) is trivial since the intersection of each countable sequence of neighborhoods of y_0 is again a neighborhood of x_0 (this is in fact the proof which is given in [1]).

To prove the proposition it is sufficient to prove (ii) for the case $z = a$. The space X_a is in fact the well-known example of the quotient of a countable union of ordinary sequences under the identification of the limit points.

We write $X_a = \mathbb{N} \times \mathbb{N} \cup \{y_0\}$. For each neighborhood V of y_0 there exists a function $f_V\colon \mathbb{N} \to \mathbb{N}$ such that V contains all pairs $\langle j,i \rangle$ for $j \geq f_V(i)$, but not the pair $\langle f_V(i)-1, i \rangle$. These conditions define f_V uniquely in terms of V, but it may happen that different neighborhoods V yield the same function.

Now let $\mathcal{V}_0$ be an uncountable collection of neighborhoods of y_0. There exists an uncountable subcollection $\mathcal{V}_1 \subseteq \mathcal{V}_0$ such that $f_V(1) = f_{V'}(1)$ for

$V,V' \in \mathcal{V}_1$. Let U_1 be an arbitrary member of $\mathcal{V}_1$.

By induction we find for $k > 1$ an uncountable subcollection $\mathcal{V}_k \subseteq \mathcal{V}_{k-1}$ such that for each pair $V,V' \in \mathcal{V}_k$ the values $f_V(j) = f_{V'}(j)$ for $j \leq k$. Again we take for U_k an arbitrary member of $\mathcal{V}_k$.

It is easy to verify that for the sequence $(U_k)_{k=1}^{\infty}$ constructed in this manner the intersection $\bigcap_{k=1}^{\infty} U_k$ again is a neighborhood of y_0. This proves assertion (ii) for X_a. □

For $z \neq a$ and $\underline{\text{cf}}(z) = a$ proposition 7 does not work. In these circumstances we use the following alternative construction: Let Y_z be the quotient space which results from the product space $S_b \times D_z$ by identifying the set $\{x_0\} \times D_z$ to a single point y_0.

PROPOSITION 8. *For $z \geq b$ the space Y_z satisfies* (i) *and* (ii).

PROOF. (i) is shown by the usual diagonalization argument, whereas (ii) again is trivial. □

Our next proposition shows that examples of non-subminispaces may be found among the sequences $S_{z,y}$ themselves.

PROPOSITION 9. *Let $\underline{\text{cf}}(z) = y$ and assume $z > 2^y$. Let y^+ be the successor cardinal of y. Then the space S_{z,y^+} satisfies* (i) *and* (ii).

PROOF. Again condition (ii) is trivial. To prove (i) we first note that $\underline{\text{cf}}(z) = y$ implies $z^y > z$ by König's theorem [2]. Furthermore, each neighborhood of x_0 is contained in at most 2^y larger neighborhoods. Hence from the assumption $z > 2^y$ we derive that $\underline{\text{lw}}(x_0)$ equals the total number of neighborhoods of x_0 and the latter equals z^y. □

Note that for the case $y = a$ the assumption $2^a = c < z$ is fulfilled for $z > a$ and $\underline{\text{cf}}(z) = a$ if CH ($b = c$) is assumed.

THEOREM 10. *For each infinite cardinal number z there exists a sequence of cardinality z which is not a subminispace.*

PROOF. Direct from propositions 7 and 8. □

REFERENCES

[1] P. van Emde Boas, *Minimally generated topologies*, in: *Topology and its Applications*, Proc. of the Int. Symp. on Topology and its Applications, Herçeg-Novi, 25-31 Aug. 1968, Beograd, 1969, pp. 146-152.

[2] E. Kamke, *Mengenlehre*, Samml. Göschen 999, Berlin, 1927.